arXiv:hep-th/0003246v2 21 Mar 2001

Integration of the SL(2,$\mathbb{R}$)/U(1) Gauged WZNW Theory by Reduction and Quantum Parafermions

C. Ford[a], G. Jorjadze[b] and G. Weigt[a]

[a]DESY Zeuthen, Platanenallee 6,

D-15738 Zeuthen, Germany

[b]Razmadze Mathematical Institute,

M.Aleksidze 1, 380093, Tbilisi, Georgia

Abstract

Using a gauge invariant reduction we directly integrate the SL(2,$\mathbb{R}$)/U(1) WZNW theory. We prove that the conserved parafermions of this theory are coset currents. Quantum mechanically, the parafermion algebra, the energy-momentum tensor, and 'auxiliary' parafermions are deformed in a self-consistent manner.

1 Introduction

Gauged Wess-Zumino-Novikov-Witten (WZNW) models are an important subclass of two-dimensional integrable conformal field theories. But there is an essential structural difference between nilpotent [1] and non-nilpotent [2] gauging. Although the equations of motion for both cases have a linear Lax pair representation [3], only the Lax pairs for nilpotently gauged (Toda) theories have been integrated directly [4].

In this paper we will focus on non-nilpotently gauged WZNW theory. In contrast to the Toda case a systematic integration method is lacking. Instead of integrating a Lax pair, we extend the methods of Lagrangian and Hamiltonian reduction in a gauge invariant manner and prove that a non-nilpotently gauged WZNW theory can be integrated directly. We rederive here the general solution of the SL(2,$\mathbb{R}$)/U(1) theory, which was found in refs [5, 6] in a non-systematic manner guided by the solution of the B_2 non-abelian Toda theory [7, 8]. While we will restrict ourselves to the simplest SL(2,$\mathbb{R}$)/U(1) case, we presume that this approach can be generalised to any gauged WZNW theory.

Gauge invariant reduction also proves that the parafermionic conserved quantities are the coset currents, and the reduced energy-momentum tensor retains the simple Sugawara form in terms of these coset currents.

The SL(2,$\mathbb{R}$)/U(1) theory is not only of mathematical interest. This model attracted much attention when it was realised that such theories have a black hole interpretation

[9]. In the early 1990's studies of the quantum SL(2,ℝ)/U(1) WZNW theory relied heavily on rather formal path integral manipulations [9, 10, 11] or related operator identities [12]. Since the attempted path integration over the $U(1)$ gauge field provided an incomplete effective action [3, 13], we define the SL(2,ℝ)/U(1) theory by the classical Lagrangian given in ref. [2]. Our goal is to perform for this theory an exact canonical quantisation, much in the same way as it has been done for Liouville theory [14, 15]. Here one recasts the general solution as a canonical transformation exchanging interacting and free fields and lifts the classical conformal transformation to an operator transformation.

We take the parafermion algebra as a starting point for quantisation. In doing this we are forced to deform the classical free field representation of the parafermions. Similar deformations have been found before in OPE based Feigin-Fuks constructions of WZNW Kac-Moody currents [16, 17, 18, 19, 20]. In addition we derive the quantum analogue of the parafermion algebra. The Sugawara form of the energy-momentum tensor motivates us to build the quantum energy-momentum tensor using only the parafermions, and we find an improvement term which in the σ-model picture could correspond to a non-perturbative dilaton. Finally, we point out that the general solution of the model has a parafermion interpretation. More precisely, the general solution contains fields of conformal weight zero related to an alternative set of 'auxiliary' parafermions which also undergo quantum deformation.

2 A Lagrangian Reformulation of the SL(2,ℝ) WZNW Theory

WZNW models are defined by the action [21, 22, 23]

$$S_{\mathrm{WZNW}}[g] = \frac{k}{8\pi} \int\limits_M h^{\mu\nu} \mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g\right)\sqrt{-h}\; d\sigma\, d\tau + k I_{\mathrm{WZ}}[g], \tag{1}$$

which includes the topological Wess-Zumino term

$$I_{\mathrm{WZ}} = \frac{1}{12\pi}\int\limits_B \mathrm{tr}\left(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\right). \tag{2}$$

Here $h_{\mu\nu} = diag(+,-)$ is the Minkowskian metric of the world surface M, h its determinant, B a volume with the boundary $\partial B = M$, k the coupling parameter, and $g(\tau,\sigma)$ is a field

which takes values in a semi-simple Lie group G. We shall restrict ourselves in this paper to the case $G = \mathrm{SL}(2,\mathbb{R})$.

To simplify the construction of the SL(2,$\mathbb{R}$)/U(1) theory it will prove useful to rewrite the topological WZNW term (2) as an integral of a local Lagrangian with global $U(1)$ symmetry. We are going to verify that the differentiation of the 2-form (for a more general treatment see [24])

$$F = \frac{2}{1 + \langle a \; g \; a \; g^{-1} \rangle} \, L_a \wedge R_a \tag{3}$$

provides the integrand of the topological WZ term (2)

$$dF = \frac{2}{3} \langle g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \rangle, \tag{4}$$

and then Stokes' theorem reduces the Wess-Zumino term to a two dimensional integral of F over $M = \partial B$. Here a is a fixed normalised time-like element of the sl(2,$\mathbb{R}$) algebra with $\langle a \; a \rangle = 1$, where $\langle \cdot \rangle = -\frac{1}{2} \mathrm{tr}(\cdot)$ denotes a normalised trace, and the left and right 1-forms of (3) are given by

$$L_a = \langle a \; dg \; g^{-1} \rangle, \qquad R_a = \langle a \; g^{-1} dg \rangle. \tag{5}$$

Let us introduce the basis of the sl(2,$\mathbb{R}$) algebra

$$t_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

It satisfies the relations

$$t_m \; t_n = -\eta_{mn} \; I + \epsilon_{mn}{}^{l} \; t_l, \tag{7}$$

where I is the unit matrix, $\eta_{mn} = \mathrm{diag}(+,-,-)$ the $3d$ Minkowskian metric, and $\epsilon_{012} = 1$. The normalised traces of the matrices t_n are then given by

$$\langle t_m \; t_n \rangle = \eta_{mn}, \qquad \langle t_l \; t_m \; t_n \rangle = \epsilon_{lmn}. \tag{8}$$

This defines an isometry between the sl(2,$\mathbb{R}$) algebra and $3d$ Minkowski space.

The left and right 1-forms

$$L_n = \langle t_n \; dg \; g^{-1} \rangle, \qquad R_n = \langle t_n \; g^{-1} \; dg \rangle \tag{9}$$

are related by $L_m = \Lambda_m{}^n(g) R_n$, where

$$\Lambda_m{}^n(g) = \langle t_m \; g \; t^n \; g^{-1} \rangle \tag{10}$$

is a Lorentz transformation matrix. Since from (9) we have $g^{-1}\, dg = t^n R_n$ and $dg\, g^{-1} = t^n L_n$, using (8) the right hand side of (4) can be written in terms of right (or left) 1-forms

$$\frac{2}{3}\langle g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\rangle = 4\ R_0 \wedge R_1 \wedge R_2. \tag{11}$$

Moreover, the differentials of (9) and (10)

$$\begin{aligned} dL_n &= \epsilon_n{}^{lm}\ L_l \wedge L_m, \qquad d\Lambda_{mn} = 2\epsilon_n{}^{kl}\ \Lambda_{mk}\ R_l \\ dR_n &= -\epsilon_n{}^{lm}\ R_l \wedge R_m, \end{aligned} \tag{12}$$

give for the left hand side of (4) the same result $4\ R_0 \wedge R_1 \wedge R_2$. This proves our statement (4). With Stokes' theorem, we finally obtain the alternative Lagrangian formulation of the WZNW theory

$$S = \int_M dz d\bar{z}\ \mathcal{L}, \qquad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{WZ}, \tag{13}$$

where the kinetic term $\mathcal{L}_0$ remains unchanged

$$\mathcal{L}_0 = -\frac{1}{\gamma^2}\ \langle g^{-1}\partial_z g\ g^{-1}\partial_{\bar{z}} g\rangle, \tag{14}$$

but the WZ term becomes

$$\mathcal{L}_{WZ} = -\frac{1}{\gamma^2}\, \frac{\langle a\ \partial_z g\ g^{-1}\rangle\langle a\ g^{-1}\partial_{\bar{z}} g\rangle - \langle a\ \partial_{\bar{z}} g\ g^{-1}\rangle\langle a\ g^{-1}\partial_z g\ \rangle}{1 + \langle a\ g\ a\ g^{-1}\rangle}. \tag{15}$$

Here we used light-cone coordinates $z = \sigma + \tau,\ \bar{z} = \tau - \sigma$, and a new coupling constant $\gamma^2 = 2\pi/k$. The Euler-Lagrange equations obtained from (13-15) reproduce consistently the dynamical equations of the WZNW theory (1)

$$\partial_{\bar{z}}(\partial_z g\ g^{-1}) = 0, \qquad \partial_z(g^{-1}\partial_{\bar{z}} g\ ) = 0. \tag{16}$$

Note that the timelike property of a ($\langle a\ a\rangle = 1$) guarantees regularity of (15).

3 The Gauged SL(2,$\mathbb{R}$)/U(1) WZNW Theory

The Lagrangian (13-15) is invariant under the global transformations

$$g \mapsto h_a(\varepsilon) g h_a(\varepsilon), \quad \text{with} \qquad h_a(\varepsilon) = e^{\varepsilon a}, \tag{17}$$

which for timelike a form the $U(1)$ subgroup of SL(2,$\mathbb{R}$).

By a standard gauging procedure we introduce the $U(1)$ gauge fields A_z, $A_{\bar{z}}$ and get the new Lagrangian

$$\mathcal{L}_G(g, A_z, A_{\bar{z}}, \partial_z g, \partial_{\bar{z}} g) = \mathcal{L}\left(g, \partial_z g - A_z(ag+ga), \partial_{\bar{z}} g - A_{\bar{z}}(ag+ga)\right), \tag{18}$$

which is invariant under the local transformations

$$A_z \mapsto A_z + \partial_z \varepsilon, \quad A_{\bar{z}} \mapsto A_{\bar{z}} + \partial_{\bar{z}} \varepsilon, \quad g \mapsto h_a(\varepsilon) g h_a(\varepsilon), \quad \varepsilon = \varepsilon(z, \bar{z}). \tag{19}$$

The non-dynamical gauge fields can easily be eliminated from (18) through their algebraic equations of motion

$$A_z = \frac{\langle a\ \partial_z g\ g^{-1}\rangle}{1 + \langle a\ g\ a\ g^{-1}\rangle}, \qquad A_{\bar{z}} = \frac{\langle a\ g^{-1}\ \partial_{\bar{z}} g\rangle}{1 + \langle a\ g\ a\ g^{-1}\rangle}. \tag{20}$$

So we obtain a gauge invariant Lagrangian only in terms of the field g

$$\begin{aligned} \mathcal{L}_G| \ &= \ -\frac{1}{\gamma^2}\left(\langle g^{-1}\partial_z g\ g^{-1}\partial_{\bar{z}} g\rangle\right. \\ &\qquad \left. - \frac{\langle a\ \partial_z g\ g^{-1}\rangle\langle a\ g^{-1}\partial_{\bar{z}} g\rangle + \langle a\ \partial_{\bar{z}} g\ g^{-1}\rangle\langle a\ g^{-1}\ \partial_z g\rangle}{1 + \langle a\ g\ a\ g^{-1}\rangle}\right). \end{aligned} \tag{21}$$

Without any loss of generality we assume $a = t_0$. Since $\langle t_0 g t_0 g^{-1}\rangle = \Lambda_{00}$ is strictly positive the denominator in (21) is never zero. It is important for the following that this Lagrangian can be rewritten entirely in terms of the gauge invariant variables

$$v_1 = \langle t_1\ g\rangle, \qquad v_2 = \langle t_2\ g\rangle. \tag{22}$$

The gauge invariance follows from $e^{\epsilon t_0}\, t_n\, e^{\epsilon t_0} = t_n$ for $n = 1, 2$. Introducing the additional variables $v_0 = \langle t_0 g\rangle$ and $c = -\langle g\rangle$, we parameterise $g(z, \bar{z}) \in \mathrm{SL}(2,\mathbb{R})$ as

$$g = cI + v^n\ t_n = \begin{pmatrix} c - v_2 & -v_1 - v_0 \\ -v_1 + v_0 & c + v_2 \end{pmatrix}, \quad \text{with} \quad c^2 + v^n v_n = 1. \tag{23}$$

Inserting this in (21), we find that the dependence on the gauge variant fields v_0 and c is eliminated, and the reduced Lagrangian becomes

$$\mathcal{L}_G| = \frac{\partial_z v_1 \partial_{\bar{z}} v_1 + \partial_z v_2 \partial_{\bar{z}} v_2}{\gamma^2(1 + v_1^2 + v_2^2)}. \tag{24}$$

This Lagrangian has a natural complex structure in terms of the Kruskal coordinates $u = v_1 + iv_2$, $\bar{u} = v_1 - iv_2$ of [9]. The resulting equation of motion

$$\partial_z \partial_{\bar{z}} u = \bar{u}\ \frac{\partial_z u\ \partial_{\bar{z}} u}{1 + u\bar{u}} \tag{25}$$

is just that obtained in [3, 5].

4 Integration of the Theory by Lagrangian Reduction

The WZNW equations of motion (16) have the well-known general solution

$$g(z,\bar z) = g_L(z)\ g_R(\bar z), \tag{26}$$

where $g_L(z)$, $g_R(\bar z)$ are arbitrary SL(2,$\mathbb{R}$) group valued (anti-)chiral functions.

Now let $g(z,\bar z)$ be a solution (26) which satisfies the conditions

$$\langle t_0\ \partial_z g_L(z)\ g_L^{-1}(z)\rangle = 0 \quad \text{and} \quad \langle t_0\ g_R^{-1}(\bar z)\ \partial_{\bar z} g_R(\bar z)\rangle = 0. \tag{27}$$

Then, due to (20), the set of functions $g(z,\bar z), A_z(z,\bar z) = A_{\bar z}(z,\bar z) = 0$ form a solution of the dynamical equations derived from (18). Since the Lagrangians (18) and (24) have in terms of the gauge invariant fields v_1 and v_2 the same dynamical equations (25), the solutions of (25) can be written as

$$u(z,\bar z) = \langle (t_1 + it_2)\ g_L(z)\ g_R(\bar z)\rangle, \tag{28}$$

where the fields g_L and g_R satisfy (27). Equations (20) and (25) imply vanishing field strength $F_{z\bar z} = \partial_z A_{\bar z} - \partial_{\bar z} A_z = 0$. Then due to gauge invariance, (28) describes the general solution of (25).

We seek these solutions in terms of unconstrained (anti-) chiral fields. Therefore, we parameterise g_L and g_R as in (23)

$$g_L(z) = c(z)I + v^n(z)t_n, \quad g_R(\bar z) = \bar c(\bar z)I + \bar v^n(\bar z)t_n, \tag{29}$$

and introduce for convenience polar coordinates for the chiral fields

$$\begin{aligned} c &= R\cos\beta, & v_0 &= R\sin\beta, \\ v_1 &= r\cos\alpha, & v_2 &= -r\sin\alpha, \end{aligned} \tag{30}$$

and similarly for the anti-chiral ones. The conditions (27) lead to $R^2\beta' - r^2\alpha' = 0$, and with $R^2 - r^2 = 1$ which follows from (23), we deduce the relations

$$R = \sqrt{\frac{\alpha'}{\alpha' - \beta'}}, \qquad r = \sqrt{\frac{\beta'}{\alpha' - \beta'}}. \tag{31}$$

Here $'$ denotes differentiation. The insertion of (29) and (30) in (28) yields finally the general solution of (25)

$$u(z,\bar{z}) = R(z)\bar{r}(\bar{z})e^{i\bar{\alpha}(\bar{z})-i\beta(z)} + r(z)\bar{R}(\bar{z})e^{-i\alpha(z)+i\bar{\beta}(\bar{z})}, \tag{32}$$

which is correctly parameterised by the two chiral and two anti-chiral functions $\alpha(z)$, $\beta(z)$, $\bar{\alpha}(\bar{z})$, $\bar{\beta}(\bar{z})$.

As a conformal field theory (24) has a traceless energy-momentum tensor $T_{z\bar{z}} = 0$, with the chiral component

$$T = T_{zz} = \frac{1}{\gamma^2}\,\frac{\partial_z\bar{u}\partial_z u}{1+u\bar{u}} = \frac{1}{\gamma^2}\,\left(\alpha'\beta' + \frac{(\alpha''\beta' - \beta''\alpha')^2}{4\alpha'\beta'(\alpha'-\beta')^2}\right), \tag{33}$$

and similarly for the anti-chiral part $\bar{T} = T_{\bar{z}\bar{z}}$. A free-field form of this energy-momentum tensor, $T(z) = {\phi_1'}^2(z) + {\phi_2'}^2(z)$, can be obtained by passing to canonical free fields ($k = 1, 2$)

$$\psi_k(\sigma,\tau) = \phi_k(z) + \bar{\phi}_k(\bar{z}), \tag{34}$$

where $\phi_k(z)$ and $\bar{\phi}_k(\bar{z})$ are chiral and anti-chiral components, respectively. The free-field transformation which solves this problem is given by

$$\begin{aligned} e^{-i\alpha} &= e^{i\gamma\phi_2}\frac{2e^{\gamma\phi_1} - e^{-\gamma\phi_1} + 2ie^{\gamma\phi_1}\Phi}{\sqrt{\left(2e^{\gamma\phi_1} - e^{-\gamma\phi_1}\right)^2 + 4e^{2\gamma\phi_1}\Phi^2}}, \\ e^{-i\beta} &= e^{i\gamma\phi_2}\frac{2e^{\gamma\phi_1} + e^{-\gamma\phi_1} - 2ie^{\gamma\phi_1}\Phi}{\sqrt{\left(2e^{\gamma\phi_1} + e^{-\gamma\phi_1}\right)^2 + 4e^{2\gamma\phi_1}\Phi^2}}\;, \end{aligned} \tag{35}$$

where $\Phi(z)$ is defined as the integral of

$$\partial_z\Phi(z) = \gamma e^{-2\gamma\phi_1(z)}\partial_z\phi_2(z). \tag{36}$$

The general solution (32) then takes the form of a canonical transformation mapping SL(2,ℝ)/U(1) fields onto free fields

$$\begin{aligned} u = e^{i\gamma(\phi_2+\bar{\phi}_2)}\Big(e^{\gamma(\phi_1+\bar{\phi}_1)}(1+\Phi\bar{\Phi}) \;&-\; \frac{1}{4}e^{-\gamma(\phi_1+\bar{\phi}_1)} \\ &+\; \frac{i}{2}(\Phi e^{\gamma(\phi_1-\bar{\phi}_1)} + \bar{\Phi}e^{-\gamma(\phi_1-\bar{\phi}_1)})\Big). \end{aligned} \tag{37}$$

In the following we will impose periodicity in the spatial direction

$$u(\sigma+2\pi,\tau) = u(\sigma,\tau), \quad \psi_k(\sigma+2\pi,\tau) = \psi_k(\sigma,\tau). \tag{38}$$

While the ψ_k are strictly periodic their chiral and anti-chiral pieces have the following monodromy

$$\phi_k(z+2\pi) = \phi_k(z) + \frac{p_k}{2}, \qquad \bar{\phi}_k(\bar{z}-2\pi) = \bar{\phi}_k(\bar{z}) - \frac{p_k}{2}, \tag{39}$$

where the p_k are momentum zero modes. With these boundary conditions the integration of (36) gives

$$\Phi(z) = -\frac{\gamma}{2\sinh(\frac{1}{2}\gamma p_1)} \int_0^{2\pi} dz'\ \partial_{z'}\phi_2(z') e^{-\frac{1}{2}\gamma p_1 \epsilon_{2\pi}(z-z') - 2\gamma\phi_1(z')}. \tag{40}$$

Inserting (40) into (37) exactly reproduces the results obtained in [6].

Thus, we have demonstrated that the gauge invariant Lagrangian reduction indeed yields an integration method which allows one to solve the equations of motion of a non-nilpotently gauged WZNW theory in a straightforward manner.

5 Integration of the Theory by Hamiltonian Reduction

The Hamiltonian reduction of the WZNW theory is an alternative method for the construction and integration of coset models. The phase space of the system (13) is given by a set of functions $R(\sigma), g(\sigma)$, where $R(\sigma)$ and $g(\sigma)$ take values in the sl(2,$\mathbb{R}$) algebra and the SL(2,$\mathbb{R}$) group, respectively. The reformulated WZNW action (13-15) can be written as $S = \int (\theta - H\, \mathrm{d}\tau)$ where the 1-form θ and the Hamiltonian H are

$$\begin{aligned} \theta &= \int_0^{2\pi} d\sigma \left(-\langle R g^{-1} \mathrm{d}g \rangle + \frac{\langle t_0 g^{-1} g' \rangle \langle t_0 \mathrm{d}g g^{-1} \rangle - \langle t_0 g' g^{-1} \rangle \langle t_0 g^{-1} \mathrm{d}g \rangle}{\gamma^2 (1 + \langle t_0\ g\ t_0\ g^{-1} \rangle)} \right) \\ H &= -\frac{1}{2} \int_0^{2\pi} d\sigma\ \left(\gamma^2\ \langle R\ R \rangle + \frac{1}{\gamma^2}\ \langle g^{-1}\ g'\ g^{-1}\ g' \rangle \right). \end{aligned} \tag{41}$$

Here $g' = \partial_\sigma g$, and d denotes the exterior derivative. Variation of $R(\sigma)$ yields the Hamiltonian equation

$$\gamma^2\ R(\sigma) = g^{-1} \partial_\tau g. \tag{42}$$

Accordingly, we parameterise the functions $R(\sigma),\ g(\sigma)$ by the SL(2,$\mathbb{R}$) group valued fields g_L and g_R

$$\begin{aligned} g(\sigma) &= g_L(\sigma) g_R(-\sigma), \\ R(\sigma) &= g_R^{-1}(-\sigma) g_L^{-1}(\sigma) g_L'(\sigma) g_R(-\sigma) + g_R^{-1}(-\sigma) g_R'(-\sigma). \end{aligned} \tag{43}$$

Then the Hamiltonian in (41) splits into chiral and anti-chiral parts $H = H_L + H_R$, where

$$H_L = -\frac{1}{\gamma^2}\int_0^{2\pi} d\sigma \ \langle g_L^{-1}\ g_L'\ g_L^{-1}\ g_L'\rangle, \tag{44}$$

and similarly for H_R. The corresponding splitting holds (up to boundary terms) also for the symplectic form $\omega = \mathrm{d}\theta$. Using (41) and (43) we obtain

$$\begin{aligned}
\omega \ = \ & -\frac{1}{\gamma^2}\int_0^{2\pi} d\sigma \ \big(\langle (g_L^{-1}\ \mathrm{d}g_L)' \wedge (g_L^{-1}\ \mathrm{d}g_L)\rangle - \langle (\mathrm{d}g_R\ g_R^{-1})' \wedge (\mathrm{d}g_R\ g_R^{-1})\rangle\big) \\
& -\frac{1}{\gamma^2}\left(\langle (g_L^{-1}\ \mathrm{d}g_L) \wedge (\mathrm{d}g_R\ g_R^{-1})\rangle - \frac{\langle t_0\, g^{-1}\,\mathrm{d}g\rangle \wedge \langle t_0\,\mathrm{d}g\, g^{-1}\rangle}{1+\langle t_0\ g\ t_0\ g^{-1}\rangle}\right)\Bigg|_0^{2\pi}.
\end{aligned} \tag{45}$$

The last term of this equation vanishes for periodic g and in this case (45) reduces to the WZNW symplectic form of [25]. Then the Hamiltonian equations split into $\dot{g}_L = g_L'$, and $\dot{g}_R = -g_R'$, providing the general solution (26). That is why we used the same notation for the g_L, g_R fields in the Hamiltonian and Lagrangian approaches.

The gauging procedure which led to the coset model (24) is equivalent to a Hamiltonian reduction with the same constraints (27). For the parameterisation (29), (30) the reduced chiral Hamiltonian and 2-form become

$$H_L| = \frac{1}{\gamma^2}\int_0^{2\pi} d\sigma\ (f'^2 + \alpha'\beta'), \qquad \omega_L| = \frac{1}{\gamma^2}\int_0^{2\pi} d\sigma\ (\mathrm{d}f' \wedge \mathrm{d}f + \mathrm{d}\beta' \wedge \mathrm{d}\alpha), \tag{46}$$

where $\tanh^2 f = \beta'/\alpha'$, and we have neglected boundary terms in ω_L. Note that the integrand of H_L is just the energy-momentum tensor (33).

A canonical free-field form of (46) can be obtained by passing to the canonical free fields ϕ_1 and ϕ_2 using again the free-field transformation (35), and we get

$$\begin{aligned}
T = \frac{1}{\gamma^2}(f'^2 + \alpha'\beta') \ &= \ \phi_1'^2 + \phi_2'^2, \\
\frac{1}{\gamma^2}(\mathrm{d}f' \wedge \mathrm{d}f + \mathrm{d}\beta' \wedge \mathrm{d}\alpha) \ &= \ \mathrm{d}\phi_1' \wedge \mathrm{d}\phi_1 + \mathrm{d}\phi_2' \wedge \mathrm{d}\phi_2 \\
& \quad + \text{boundary terms}.
\end{aligned} \tag{47}$$

In fact the free-field transformation (35) was obtained as a solution of these equations.

This shows that the Hamiltonian reduction like the Lagrangian reduction provides a convenient approach for the integration of our non-nilpotently gauged SL(2,$\mathbb{R}$)/U(1) WZNW theory. We envisage that these methods should be generalisable to other gauged WZNW theories.

6 The Parafermionic SL(2,$\mathbb{R}$)/U(1) Coset Currents

It is well known that the chiral WZNW currents

$$\gamma^2 J_k(z) = \langle t_k \partial_z g(z,\bar{z})\ g^{-1}(z,\bar{z})\rangle = \langle t_k \partial_z g_L(z)\ g_L^{-1}(z)\rangle \tag{48}$$

satisfy the linear Kac-Moody algebra

$$\{J_k(z), J_l(z')\} = \epsilon_{kl}{}^m J_m(z)\delta(z-z') + \frac{1}{2\gamma^2}\eta_{kl}\delta'(z-z'). \tag{49}$$

Here we would like to understand how these properties are impacted by the reduction. The chiral currents

$$\gamma^2 V_\pm(z) = \langle (t_1 \pm i t_2)\partial_z g_L(z)\ g_L^{-1}(z)\rangle \tag{50}$$

are of particular interest. Taking into account the parameterisations (29,30), the constraints (31) and the free-field transformations (35), a straightforward calculation yields

$$V_\pm(z) = \frac{1}{\gamma}\Big(\partial_z\phi_1(z) \pm i\partial_z\phi_2(z)\Big)e^{\pm 2i\gamma\phi_2(z)}. \tag{51}$$

Interestingly, these fields are the free-field transformed conserved parafermions of [2, 5], which now obtain a coset current interpretation through the reduction.

The Sugawara energy-momentum tensor of the WZNW theory

$$T(z) = -\gamma^2 J^k(z)J_k(z) \tag{52}$$

retains this simple form in terms of the parafermionic coset currents [2, 5] even after reduction

$$T(z) = \gamma^2 V_+(z)V_-(z). \tag{53}$$

The free-field parameterisation of the parafermions (51) clearly leads to the free-field energy-momentum tensor (47). But in contrast to the WZNW Kac-Moody currents (48), the coset currents satisfy a non-linear and non-local Poisson bracket algebra, which can be calculated either through the Dirac bracket method or directly from the free field representations. The results for the Poisson brackets depend on the chosen boundary conditions. Here we work with the periodic ones prescribed in section 3. For the chiral and anti-chiral components $\phi_k(z)$, $\bar{\phi}_k(\bar{z})$ of the canonical free fields $\psi_k(\sigma,\tau)$ (34) we choose the usual mode

expansion

$$
\begin{aligned}
\phi_k(z) &= \frac{1}{2}q_k + \frac{1}{4\pi}p_k z + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{a_n^{(k)}}{n}\mathrm{e}^{-inz}, \\
\bar{\phi}_k(\bar{z}) &= \frac{1}{2}q_k + \frac{1}{4\pi}p_k \bar{z} + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{\bar{a}_n^{(k)}}{n}\mathrm{e}^{-in\bar{z}},
\end{aligned} \tag{54}
$$

with the canonical mode algebra

$$
\begin{aligned}
\{q_k, p_l\} &= \delta_{k,l}, \quad i\left\{a_n^{(k)}, a_n^{(l)}\right\} = m\delta_{k,l}\delta_{m+n,0}, \\
\left\{q_k, a_m^{(l)}\right\} &= 0, \qquad\quad \left\{p_k, a_m^{(l)}\right\} = 0.
\end{aligned} \tag{55}
$$

Instead of the $V_\pm(z)$'s we will consider the periodic parafermions [6]

$$
W_\pm(z) = \frac{1}{\gamma}\Big(\partial_z\phi_1(z) \pm i\partial_z\phi_2(z)\Big)e^{\pm 2i\gamma\varphi_2(z)}, \tag{56}
$$

where φ_2 is ϕ_2 with the momentum zero mode p_2 removed and the whole q_2 zero mode of the free field (34) included, i.e. $\varphi_2(z) = \frac{1}{2}q_2 + \phi_2(z)|_{p_2=0}$. These periodic coset currents have the algebra

$$
\begin{aligned}
\{W_\pm(z), W_\pm(z')\} &= \gamma^2 W_\pm(z)W_\pm(z')\,h(z-z'), \\
\{W_\pm(z), W_\mp(z')\} &= -\gamma^2 W_\pm(z)W_\mp(z')\,h(z-z') \\
&\quad + \frac{1}{\gamma^2}\left(\partial_z + \frac{i\gamma p_2}{2\pi}\right)\delta_{2\pi}(z-z'), \\
\{p_2, W_\pm(z')\} &= \mp 2i\gamma W_\pm(z'),
\end{aligned} \tag{57}
$$

where

$$
h(z) = \left(\epsilon_{2\pi}(z) - \frac{z}{\pi}\right) \tag{58}
$$

is the periodic sawtooth function and $\epsilon_{2\pi}(z)$ the stairstep function[1]. Note that the momentum zero mode p_2 enters into the periodic parafermion algebra.

The energy-momentum tensor (53), now expressed in terms of the $W_\pm$'s, provides the Virasoro algebra

$$
\{T(z), T(z')\} = -\partial_{z'}T(z')\delta_{2\pi}(z-z') + 2T(z')\partial_z\delta_{2\pi}(z-z'), \tag{59}
$$

[1] $\epsilon_{2\pi}(\sigma) = 2n+1$ for $2n\pi < \sigma < (2n+2)\pi$ which coincides with $\mathrm{sign}(\sigma)$ for $-2\pi < \sigma < 2\pi$.

indicating for it conformal weight *two*, and the parafermions $W_\pm(z)$

$$\begin{aligned} \{T(z), W_\pm(z')\} &= -\partial_{z'} W_\pm(z')\delta_{2\pi}(z-z') + W_\pm(z')\partial_z\delta_{2\pi}(z-z') \\ &\mp \frac{i\gamma p_2}{2\pi} W_\pm(z')\delta_{2\pi}(z-z') \end{aligned} \tag{60}$$

have conformal weight *one*. Finally, we add a useful formula which generates the energy-momentum tensor $T(z)$ through a Poisson bracket

$$\begin{aligned} \{D_z W_+(z), W_-(z')\} &= \gamma^2 D_z W_+(z) W_-(z') h(z-z') \\ &-2T(z)\delta_{2\pi}(z-z') + \frac{1}{\gamma^2} D_z^2 \delta_{2\pi}(z-z'), \end{aligned} \tag{61}$$

where

$$D_z = \partial_z + \frac{i\gamma p_2}{2\pi}. \tag{62}$$

Equation (61) becomes important quantum mechanically because the operator product $W_+(z)W_-(z)$ is ill defined and cannot be used to define a $T(z)$ operator.

7 Canonical Quantisation of the Parafermions

In this section we shall determine explicitly the quantum analogue of the parafermion algebra (57) instead of analysing operator product expansions. The quantisation of the theory will be defined by replacing Poisson brackets of the canonical free fields by commutators $i\hbar\{\ ,\ \} \to [\ ,\ ]$, and non-linear expressions in the free fields will be normal ordered. But calculations with normal ordered operators usually yield anomalous contributions. Such anomalies can be avoided by quantum mechanically deforming the composite operators of the theory. Therefore, let us define the normal ordered parafermion operators corresponding to (56) as

$$W_\pm(z) = \frac{1}{\gamma} : \Big(\eta\partial_z\phi_1(z) \pm i\partial_z\phi_2(z)\Big) e^{\pm 2i\gamma\varphi_2(z)} :, \tag{63}$$

where η is a deformation parameter with the classical limit $\eta = 1$.

First we look for the quantum analogue of the Poisson brackets (57), starting with the simplest example $\{W_+(z), W_+(z')\} = \gamma^2 W_+(z)W_+(z')h(z-z')$. In the appendix we have explicitly derived the relation

$$\begin{aligned} \frac{W_+(z)W_+(z')}{e^{i\hbar\gamma^2 h^+(z-z')}} &- \frac{W_+(z')W_+(z)}{e^{-i\hbar\gamma^2 h^-(z-z')}} = \\ &\frac{i\hbar}{2\gamma^2}\left(\eta^2 - 1 + \frac{\gamma^2\hbar}{\pi}\right) : e^{2i\gamma\varphi_2(z)} e^{2i\gamma\varphi_2(z')} : \partial_z\delta_{2\pi}(z-z'), \end{aligned} \tag{64}$$

where

$$h^{\pm}(z) = \frac{1}{2}h(z) \mp \frac{i}{2\pi}\log\left(4\sin^2\frac{z}{2}\right) \tag{65}$$

are the positive and negative frequency parts of $h(z)$, respectively. The right hand side of (64) is proportional to the operator $: e^{2i\gamma\varphi_2(z)}e^{2i\gamma\varphi_2(z')} :$ which evidently cannot be rewritten bilocally in terms of the parafermions, as is necessary to have a closed operator algebra. However we can remove the offending term altogether by imposing the restriction

$$\eta^2 - 1 + \frac{\gamma^2\hbar}{\pi} = 0, \qquad \text{or} \qquad \eta = \pm\sqrt{1 - \frac{\hbar\gamma^2}{\pi}}. \tag{66}$$

The classical limit $\eta = 1$ corresponds to the positive square root. With this choice we have

$$\frac{W_+(z)W_+(z')}{e^{i\hbar\gamma^2 h^+(z-z')}} - \frac{W_+(z')W_+(z)}{e^{-i\hbar\gamma^2 h^-(z-z')}} = 0. \tag{67}$$

The quantum relations corresponding to the other Poisson brackets of (57) are (see the appendix)

$$\begin{aligned}
\frac{W_-(z)W_-(z')}{e^{i\hbar\gamma^2 h^+(z-z')}} - \frac{W_-(z')W_-(z)}{e^{-i\hbar\gamma^2 h^-(z-z')}} &= 0, && (68)\\
\frac{W_+(z)W_-(z')}{e^{-i\hbar\gamma^2 h^+(z-z')}} - \frac{W_-(z')W_+(z)}{e^{i\hbar\gamma^2 h^-(z-z')}} &= \frac{i\hbar}{\gamma^2}\left(\partial_z + \frac{i\gamma p_2}{2\pi}\right)\delta_{2\pi}(z-z'), && (69)\\
[p_2, W_\pm(z)] &= \pm 2\hbar\gamma W_\pm(z). && (70)
\end{aligned}$$

As in the derivation of (67) it is necessary to impose (66) to eliminate anomalous contributions.

To check that these operator relations correspond to the classical Poisson brackets we expand the exact formulae in powers of $\hbar$, e.g. equation (67) gives

$$[W_+(z), W_+(z')] - i\hbar\gamma^2 W_+(z)W_+(z')h(z-z') + O(\hbar^2) = 0. \tag{71}$$

Here we have used the splitting relation $h(z) = h^+(z) + h^-(z)$, which follows immediately from (65).

8 The Energy-Momentum Tensor Operator

As was mentioned before, to generate the quantum energy-momentum tensor we should consider the quantum analogue of (61). From the calculations given at the end of the

appendix it follows that

$$\frac{D_zW_+(z)W_-(z')}{e^{-i\hbar\gamma^2h^+(z-z')}} - \frac{W_-(z')D_zW_+(z)}{e^{i\hbar\gamma^2h^-(z-z')}} = \frac{i\hbar}{\gamma^2}\left(1+\frac{\hbar\gamma^2}{2\pi}\right)D_z{}^2\delta_{2\pi}(z-z') \tag{72}$$
$$-2i\hbar\eta^2\Bigg(:(\partial_z\phi_1)^2(z'):+:(\partial_z\phi_2)^2(z'):$$
$$+\frac{\hbar\gamma}{2\pi\eta}\partial_z^2\phi_1(z')+\frac{\hbar\gamma^2}{(4\pi\eta)^2}\Bigg)\delta_{2\pi}(z-z').$$

The second entry on the right hand side just corresponds to the term $-2T(z')\delta_{2\pi}(z-z')$ of the classical Poisson bracket (61) which suggests that

$$T(z) =: (\partial_z\phi_1)^2(z):+:(\partial_z\phi_2)^2(z):+\frac{\hbar\gamma}{2\pi\eta}\partial_z^2\phi_1(z)+\frac{\hbar\gamma^2}{(4\pi\eta)^2} \tag{73}$$

is the free-field energy-momentum tensor of our model with an additional improvement term. In the σ-model interpretation this improvement term could correspond to a non-perturbative dilaton [9, 10, 11, 12].

Note that the energy-momentum tensor obeys the Virasoro algebra

$$\begin{aligned}[T(z),T(z')] &= -i\hbar\partial_{z'}T(z')\delta_{2\pi}(z-z')+2i\hbar T(z')\partial_z\delta_{2\pi}(z-z') \\ &\quad -\frac{i\hbar c}{24\pi}(\partial_z^3+2\partial_z)\delta_{2\pi}(z-z'),\end{aligned} \tag{74}$$

with central charge

$$c=\hbar\left(2+\frac{3\gamma^2}{\pi\eta^2}\right), \tag{75}$$

in agreement with the results of [9, 26].

Classically the parafermions are primary fields of weight one. Quantum mechanically the commutator

$$\begin{aligned}[T(z),W_+(z')] &= i\hbar\left(1+\frac{\hbar\gamma^2}{2\pi}\right)W_+(z')\partial_z\delta_{2\pi}(z-z') \\ &\quad -i\hbar\partial_{z'}W_+(z')\delta_{2\pi}(z-z')+\frac{\hbar\gamma}{2\pi}:p_2W_+(z'):\delta_{2\pi}(z-z')\end{aligned} \tag{76}$$

shows that the quantum parafermions have the shifted conformal weight $1+\hbar\gamma^2/(2\pi)$.

9 Auxiliary Parafermions

In order to quantise the canonical transformation (37) we need the operators corresponding to the simple exponentials in (37) as well as the non-local fields $\Phi(z)$ and $\bar\Phi(\bar z)$. However,

it turns out that

$$A(z) = -\frac{1}{2}e^{-2\gamma\phi_1(z)} - i\Phi(z), \qquad \bar{A}(\bar{z}) = -\frac{1}{2}e^{-2\gamma\phi_1(\bar{z})} - i\Phi(\bar{z}) \tag{77}$$

are more amenable to a quantum treatment. Actually $A(z)$ is one of the chiral fields which parametrises the general solution given in [5]. The derivative of $A(z)$ has a similar structure to that of the parafermions, and so we will refer to $A'(z)$ as an auxiliary parafermion

$$\tilde{V}_-(z) = \gamma\Big(\partial_z\phi_1(z) - i\partial_z\phi_2(z)\Big)e^{-2\gamma\phi_1(z)}. \tag{78}$$

It satisfies a closed chiral algebra if we replace $\phi_1(z)$ by $\frac{1}{2}q_1+\phi_1(z)$. The somewhat artificial doubling of the q_1 zero-mode is an artifact of our strict separation of chiral and anti-chiral objects, whereas each term in the general solution is a product of chiral and anti-chiral pieces. Following the recipe of section 7 the quantum auxiliary parafermion turns out to be

$$\tilde{V}_-(z) = \gamma : \Big(\eta\partial_z\phi_1(z) - i\partial_z\phi_2(z)\Big)e^{-2\gamma\eta^{-1}(\frac{1}{2}q_1+\phi_1(z))} : . \tag{79}$$

It commutes with $V_-(z')$, but unlike $V_-(z)$ retains its classical conformal weight *one*. Thus, $A(z)$ has conformal weight zero, suggesting that it plays the role of a screening charge.

Summarising, only one fixed deformation parameter is sufficient for consistent quantisation. Of course, our quantisation of the SL(2,ℝ)/U(1) theory is still incomplete. In particular, it remains to quantise the black hole metric. It could answer the question whether a non-perturbative dilaton renders this metric dynamical.

Acknowledgments

G.W. thanks J. Schnittger for discussions on the quantum aspects of the problem. We would like to thank C. J. Biebl for reading the manuscript. G.J. is grateful to DESY Zeuthen for hospitality. His research was supported by grants from the DFG, GSRT and RFBR (99-01-00151).

A Normal Ordered Operator Identities

In this appendix we elaborate on the normal ordered operator identities quoted in the text. To effect the normal ordering we decompose the chiral free fields $\phi_i(z)$ defined in (54) as

follows

$$\phi_i(z) = \frac{1}{2}q_i + \frac{1}{4\pi}p_i z + \phi_i^+(z) + \phi_i^-(z), \tag{80}$$

where

$$\phi_i^\pm(z) = \pm\frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{a_{\pm n}^{(i)}}{n}e^{\mp inz}. \tag{81}$$

$\phi_i^-(z)$ and $\phi_i^+(z)$ will be interpreted as creation and annihilation operators, respectively. The equivalent anti-chiral constructions will not be considered here.

Using the commutator algebra

$$[q_i, p_j] = i\hbar\delta_{i,j}, \qquad [a_m^{(i)}, a_n^{(j)}] = m\hbar\delta_{i,j}\delta_{m+n,0}, \quad i,j = 1,2 \tag{82}$$

it follows that

$$[\phi_i^\pm(z), \phi_j^\pm(z')] = 0, \qquad [\phi_i^\pm(z), \phi_j^\mp(z')] = -\frac{i\hbar}{4}\delta_{ij}h^\pm(z-z'), \tag{83}$$

where

$$h^\pm(z) = \epsilon^\pm(z) - \frac{z}{2\pi}. \tag{84}$$

Here the $\epsilon^\pm(z)$ denote the positive and negative frequency parts of the stairstep function $\epsilon_{2\pi}(z)$, and have the Fourier series representation

$$\epsilon^+(z) = \frac{z}{2\pi} + \frac{i}{\pi}\sum_{n>0}\frac{e^{-in(z-i\varepsilon)}}{n}, \qquad \epsilon^-(z) = \frac{z}{2\pi} + \frac{i}{\pi}\sum_{n<0}\frac{e^{-in(z+i\varepsilon)}}{n}. \tag{85}$$

Note that we have included a convergence factor, $\varepsilon > 0$. The $\epsilon^\pm(z)$ functions have the properties $\epsilon^{+*}(z) = \epsilon^-(z)$, $\epsilon^+(-z) = -\epsilon^-(z)$, and in the limit $\varepsilon \to 0$ they exhibit free-field short distance singularities

$$\epsilon^\pm(z) = \frac{1}{2}\epsilon_{2\pi}(z) \mp \frac{i}{2\pi}\log\left(4\sin^2\frac{z}{2}\right). \tag{86}$$

We also introduce 'split' delta functions $\delta^+(z) = 1/2\,\partial_z\epsilon^+(z)$ [27]

$$\delta^+(z) = \frac{1}{4\pi} + \frac{1}{2\pi}\sum_{n>0}e^{-in(z-i\varepsilon)} = -\frac{1}{4\pi} + \frac{1}{2\pi}\frac{1}{1-e^{-i(z-i\varepsilon)}}, \tag{87}$$

and similarly $\delta^-(z) = 1/2\,\partial_z\epsilon^-(z)$, which have the properties $\delta^{+*}(z) = \delta^-(z)$, $\delta^+(-z) = \delta^-(z)$, and as $\varepsilon \to 0$ the $\delta^\pm(z)$ sum up to the periodic delta function $\delta_{2\pi}(z)$.

Now we are ready to establish the normal ordered operator identities quoted in the text. As usual normal ordering moves creation and annihilation operators respectively to the left and right, and the Hermitian normal ordering of zero modes $:e^{2q}f(p):=e^q f(p)e^q$ will be understood. With the definitions

$$e_\pm(z):=e^{i\gamma q_2}e^{2i\gamma\phi_2^\pm(z)},\quad \nu(z):=\frac{1}{\gamma}\Big(\eta\partial_z\phi_1(z)+i\partial_z\phi_2(z)\Big), \tag{88}$$

our periodic parafermion operator $W_+(z)$ (63) can be written

$$W_+(z)=e_-(z)\nu(z)e_+(z). \tag{89}$$

Let us start with a quantum analogue of the Poisson bracket $\{W_+(z),W_+(z')\}$. Naively one would consider the commutator, $[W_+(z),W_+(z')]$, suggesting that we should compute the operator product $W_+(z)W_+(z')$. Using the identity $e^Ae^B=e^Be^Ae^{[A,B]}$, which holds if $[A,B]$ commutes with A and B, we have $e_+(z)e_-(z')=e_-(z')e_+(z)e^{i\hbar\gamma^2h^+(z-z')}$, so that

$$e^{-i\hbar\gamma^2h^+(z-z')}W_+(z)W_+(z')=e_-(z)\nu(z)e_-(z')e_+(z)\nu(z')e_+(z'). \tag{90}$$

But this operator is still not correctly normal ordered. We decompose $\nu(z)$ as in (80)

$$\begin{aligned}\nu(z)&=\nu^+(z)+\nu^-(z)+\frac{\eta p_1+ip_2}{4\pi\gamma},\\ \nu^\pm(z)&=\frac{1}{\gamma}\left(\eta\partial_z\phi_1^\pm(z)+i\partial_z\phi_2^\pm(z)\right).\end{aligned} \tag{91}$$

With a little algebra the right hand side of (90) can be rewritten as follows

$$\begin{aligned}e^{-i\hbar\gamma^2h^+(z-z')}\,W_+(z)W_+(z')\ &=\ :W_+(z)W_+(z'): \\ &\quad+e_-(z)e_-(z')[\nu^+(z),\nu^-(z')]e_+(z)e_+(z')\\ &\quad+e_-(z)[\nu(z),e_-(z')]\nu(z')e_+(z)e_+(z')\\ &\quad+e_-(z)e_-(z')\nu(z)[e_+(z),\nu(z')]e_+(z')\\ &\quad+e_-(z)[\nu(z),e_-(z')][e_+(z),\nu(z')]e_+(z').\end{aligned} \tag{92}$$

Using the results

$$\begin{aligned}[\nu(z),e_\mp(z')]&=i\hbar e_\mp(z')\delta^\pm(z-z'),\\ [\nu^+(z),\nu^-(z')]&=\frac{i\hbar}{2\gamma^2}(\eta^2-1)\partial_z\delta^+(z-z'),\end{aligned} \tag{93}$$

we have

$$\frac{W_+(z)W_+(z')}{e^{i\hbar\gamma^2 h^+(z-z')}} = \,:W_+(z)W_+(z'):$$
$$+i\hbar:\left(e^{2i\gamma\varphi_2(z)}W_+(z') - W_+(z)e^{2i\gamma\varphi_2(z')}\right):\delta^+(z-z')$$
$$+:e^{2i\gamma\varphi_2(z)}e^{2i\gamma\varphi_2(z')}:\left(\frac{i\hbar}{2\gamma^2}(\eta^2-1)\partial_z\delta^+(z-z')\right.$$
$$\left. + \hbar^2\left(\delta^+(z-z')\right)^2\right). \tag{94}$$

At this point the following identity is useful

$$\left[\delta^+(z)\right]^2 = \frac{1}{(4\pi)^2} + \frac{i}{2\pi}\partial_z\delta^+(z), \tag{95}$$

which is valid even for finite ε. Using this formula the right hand side of (90) can be written linearly in $\delta^+(z-z')$ and its derivative. Recall that this distribution becomes $\delta^-(z-z')$ on exchanging z and z'. Thus, if we take (94) and subtract the equation obtained by exchanging z and z', we get

$$\frac{W_+(z)W_+(z')}{e^{i\hbar\gamma^2 h^+(z-z')}} - \frac{W_+(z')W_+(z)}{e^{-i\hbar\gamma^2 h^-(z-z')}} =$$
$$i\hbar:\left(e^{2i\gamma\varphi_2(z)}W_+(z') - W_+(z)e^{2i\gamma\varphi_2(z')}\right):\delta_{2\pi}(z-z')$$
$$+\frac{i\hbar}{2\gamma^2}\left(\eta^2-1+\frac{\hbar\gamma^2}{\pi}\right):e^{2i\gamma\varphi_2(z)}e^{2i\gamma\varphi_2(z')}:\partial_z\delta_{2\pi}(z-z'). \tag{96}$$

The first term on the right hand side is zero since the prefactor of $\delta_{2\pi}(z-z')$ tends to zero as $z \to z'$, and so (64) follows immediately.

Since all the other results quoted in the text can be derived by the same technique, we will be rather sketchy from now on. Evaluating the operator products $W_+(z)W_-(z')$ and $W_-(z')W_+(z)$ one is led to

$$\frac{W_+(z)W_-(z')}{e^{-i\hbar\gamma^2 h^+(z-z')}} - \frac{W_-(z')W_+(z)}{e^{i\hbar\gamma^2 h^-(z-z')}} =$$
$$i\hbar:\left(\nu(z)-\nu^*(z')\right)e^{2i\gamma\varphi_2(z)}e^{-2i\gamma\varphi_2(z')}:\delta_{2\pi}(z-z')$$
$$+\frac{i\hbar}{2\gamma^2}\left(\eta^2+1+\frac{\hbar\gamma^2}{\pi}\right):e^{2i\gamma\varphi_2(z)}e^{-2i\gamma\varphi_2(z')}:\partial_z\delta_{2\pi}(z-z'). \tag{97}$$

Using the identity $f(z)\delta_{2\pi}(z-z') = f(z')\delta_{2\pi}(z-z')$ (valid for periodic $f(z)$) and derivatives thereof

$$\frac{W_+(z)W_-(z')}{e^{-i\hbar\gamma^2 h^+(z-z')}} - \frac{W_-(z')W_+(z)}{e^{i\hbar\gamma^2 h^-(z-z')}} = -\frac{\hbar p_2}{2\pi\gamma}\delta_{2\pi}(z-z') \tag{98}$$

$$+\frac{\hbar}{\gamma}\left(\eta^2-1+\frac{\hbar\gamma^2}{\pi}\right)\partial_z\varphi_2(z')\delta_{2\pi}(z-z')$$
$$+\frac{i\hbar}{2\gamma^2}\left(\eta^2+1+\frac{\hbar\gamma^2}{\pi}\right)\partial_z\delta_{2\pi}(z-z').$$

(69) follows on imposing (66).

We conclude by laying the ground for the derivation of the energy-momentum operator. To prove (72) one first establishes that

$$\begin{aligned}\frac{D_zW_+(z)W_-(z')}{e^{-i\hbar\gamma^2h^+(z-z')}}-\frac{W_-(z')D_zW_+(z)}{e^{i\hbar\gamma^2h^-(z-z')}} &= \\ -2i\hbar\gamma^2A(z,z') &+ \frac{i\hbar}{\gamma^2}D_z^2\delta_{2\pi}(z-z'),\end{aligned} \tag{99}$$

where

$$A(z,z')=\frac{W_+(z)W_-(z')}{e^{-i\hbar\gamma^2h^+(z-z')}}\delta^+(z-z')+\frac{W_-(z')W_+(z)}{e^{i\hbar\gamma^2h^-(z-z')}}\delta^-(z-z'). \tag{100}$$

In computing $A(z,z')$ we encounter the same operator products as in the calculation of (97). The result can be written as

$$\begin{aligned}A(z,z') &= \left(:W_+(z)W_-(z'):+\frac{\hbar}{16\pi^2}\right)\delta_{2\pi}(z-z') \\ &-\frac{\hbar}{2\pi}:e^{2i\gamma\varphi_2(z)}e^{-2i\gamma\varphi_2(z')}\Big(\nu(z)-\nu^*(z')\Big):\partial_z\delta_{2\pi}(z-z') \\ &-\frac{\hbar}{8\pi\gamma^2}\left(\eta^2+1+\frac{\hbar\gamma^2}{\pi}\right):e^{2i\gamma\varphi_2(z)}e^{-2i\gamma\varphi_2(z')}:\partial_z^2\delta_{2\pi}(z-z').\end{aligned} \tag{101}$$

Again we used (95) and its derivative $\delta^+(z)\partial_z\delta^+(z)=i\partial_z^2\delta^+(z)/(4\pi)$. With the help of (66) as well as standard properties of the delta function (101) reduces to

$$\begin{aligned}A(z,z') = & \left[\frac{\eta^2}{\gamma^2}\Big(:(\partial_z\phi_1)^2(z'):+:(\partial_z\phi_2)^2(z'):\Big)+\frac{\hbar\eta}{2\pi\gamma}\partial_z^2\phi_1(z')\right. \\ & \left.+\frac{\hbar}{16\pi^2}-\frac{\hbar}{4\pi\gamma^2}\left(\partial_z+\frac{i\gamma p_2}{2\pi}\right)^2\right]\delta_{2\pi}(z-z').\end{aligned} \tag{102}$$

Inserting this into (99) gives (72).

References

[1] P. Forgacs, A. Wipf, J. Balog, L. Feher, L. O'Raifeartaigh, *Phys.Lett.* **B227** (1989) 214.

[2] K. Bardakci, M. Crescimanno, E. Rabinovici, *Nucl. Phys.* **B344** (1990) 344.

[3] U. Müller, G. Weigt, *Comm. Math. Phys.* **205** (1999) 421.

[4] A. N. Leznov, M. V. Saveliev, *Acta Appl. Math.* **16** (1989) 1.

[5] U. Müller, G. Weigt, *Phys. Lett.* **B400** (1997) 21.

[6] U. Müller, G. Weigt, *Nucl. Phys.* **B568** (2000) 457.

[7] J.-L. Gervais, M. V. Saveliev, *Phys.Lett.* **B286** (1992) 271.

[8] A. Bilal, *Nucl. Phys.* **B422** (1994) 258.

[9] E. Witten, *Phys. Rev.* **D44** (1991) 314.

[10] I. Bars, K. Sfetsos, **D48** (1993) 844.

[11] A. A. Tseytlin, *Nucl. Phys.* **B399** (1993) 601.

[12] R. Dijkgraaf, E. Verlinde, H. Verlinde, *Nucl. Phys.* **B371** (1992) 269.

[13] T. H. Buscher, *Phys. Lett.* **B201** (1988) 466.

[14] E. Braaten, T Curtright, and C. Thorn, *Ann. Phys.* **147** (1983) 365.

[15] H.J. Otto and G. Weigt, *Z. Phys.* **C31** (1986) 219.

[16] A. B. Zamolodchikov and V. A. Fateev, *Sov. Phys. JETP* **62** (1985) 215.

[17] A. Gerasimov, A. Marshakov and A. Morozov, *Nucl. Phys.* **B328** (1989) 664.

[18] T. Jayaraman, K. S. Narain and M. H. Sarmadi, *Nucl. Phys.* **B343** (1990) 418.

[19] D. Nemeschansky, *Phys. Lett.* **B224** (1989) 121.

[20] I. Bakas and E. Kiritsis, *Int. J. Mod. Phys.* **A7** (1992) 55.

[21] J. Wess, B. Zumino, *Phys. Lett.* **B37** (1971) 95.

[22] S. P. Novikov, *Usp. Math. Nauk.* **37** (1982) 3.

[23] E. Witten, *Comm. Math. Phys.* **92** (1984) 455.

[24] C. Ford, G. Jorjadze, G. Weigt, *Turk J Phy*, **24** (2000) 311.

[25] M.F. Chu, P. Goddard, I Halliday, D. Olive and A Schwimmer, *Phys. Lett.*, **B266**, (1991) 71.

[26] J. Lykken, *Nucl. Phys.* **B313** (1989) 473.

[27] C. Ford and L. O'Raifeartaigh, *Nucl. Phys.* **B460** (1996) 203.